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SOME GEOMETRIC PROPERTIES OF GOTZMANN COEFFICIENTS

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Abstract. In this paper, we study how the Hilbert polynomial, associated with a reduced closed subscheme X of codimension 2 in $\mathbb{P}^N$, reveals geometric information about X. Although it is known that the Hilbert polynomial can tell us about the scheme's degree and arithmetic genus, we find additional geometric information it can provide for smooth varieties of codimension 2. To do this, we introduce the concept of Gotzmann coefficients, which helps to extract more information from the Hilbert polynomial. These coefficients are based on the binomial expansion of values of the Hilbert function. Our method involves combining techniques from initial ideals and partial elimination ideals in a novel way. We show how these coefficients can determine the degree of certain geometric features, such as the singular locus appearing in a generic projection, for smooth varieties of codimension 2.

1. Introduction

Let X be an closed subscheme in $\mathbb{P}^N$ with its defining ideal I_X in the homogeneous coordinate ring $R = k[x_0, \dots, x_N]$ of $\mathbb{P}^N$, where k is the algebraically closed field of characteristic 0. Since X is the intersection of several hypersurfaces, we ask how many linearly independent hypersurfaces of degree n cut out X. This boils down to calculating the vector space dimension of $(I_X)_n$, which also means finding the dimension as a k-vector space at degree n in the homogeneous coordinate ring R/I_X. The Hilbert function of X, which is denoted by H_X and also referred to as H_{R/I_X}, is the numerical function defined as

$$H_X(n) = \dim_k(R/I_X)_n.$$

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In his seminal paper [14], Hilbert proved that R/I_X always has the minimal free resolution of finite length. This foundational result allows us to write the Hilbert function $H_X(n)$ as a polynomial $P_X(n)$ in terms of n for sufficiently large values of n. This polynomial $P_X(n)$ is referred to as the Hilbert polynomial of X.

Exploring the geometric information encoded in the Hilbert function or Hilbert polynomial is an intriguing topic, and numerous studies from various perspectives have been conducted in this regard. In this paper, we will focus on the growth of the Hilbert function to achieve our main result (see Theorem 3.3). A foundational result concerning the growth of the Hilbert function was provided by Macaulay in [8, 15, 18], who explored the growth of the Hilbert function for lex-segment ideals and gave an upper bound in terms of the binomial expansion of its values. Subsequent researchers have interpreted the extremal behavior of the Hilbert function geometrically. That is, their efforts have been made to explain the geometric properties of the linear system given by the ideal component at a specific degree when maximal growth is achieved [1, 2, 3, 4, 7, 9, 10, 11, 17].

In this context, our paper investigates the geometric information conveyed by the maximal growth of the Hilbert polynomial. We express the Hilbert polynomial as a sum of binomials using Gotzmann Regularity Theorem [12] and introduce the concept of Gotzmann coefficients. Additionally, by using Gröbner bases theory in generic coordinates, we show that for a smooth projective variety X of codimension 2 in $\mathbb{P}^N$, the Hilbert polynomial discloses details about the degree of the singular locus of the image under generic projections.

2. Gotzmann coefficients of Hilbert polynomials

To explain the main result of this paper, it is necessary to introduce some basic definitions and notations. We begin by defining the binomial expansion of natural numbers.

Definition 2.1. Let $n > 0$ and $c > 0$ be positive integers. The *n-th binomial expansion* of c is the unique expression

$$c = \binom{k_n}{n} + \binom{k_{n-1}}{n-1} + \cdots + \binom{k_\delta}{\delta},$$

where $k_n > k_{n-1} > \cdots > k_\delta \geq \delta > 0$. We introduce the notation:

$$c^{\langle n\rangle} = \binom{k_n+1}{n+1} + \binom{k_{n-1}+1}{n} + \cdots + \binom{k_\delta+1}{\delta+1}.$$

Macaulay showed the following result concerning the growth of the Hilbert function.

Theorem 2.2 (Macaulay's Theorem). *A numerical function $H : \mathbb{N} \to \mathbb{N}$ is the Hilbert function of some homogeneous ideal $I \subset R$ if and only if*

$$H_{R/I}(n+1) \leq H_{R/I}(n)^{<n>} \text{ for all } n > 0.$$

This result includes combinatorial aspects, and for a survey of some related results, see [8, 15, 18]. Theorem 2.2 informs us what the maximal growth of the Hilbert function is. For a given algebraic scheme X, results on the geometric characteristics of the linear system of hypersurfaces of degree n defined by $(I_X)_n$ when the Hilbert function achieves maximal growth at a certain degree n can be found in [1, 2, 3, 4].

Every Hilbert function will always achieve maximal growth after a sufficiently large degree. This fact was shown by Gotzmann. He was interested in knowing the bounds of regularity for homogeneous ideals with a given Hilbert function, and he showed that the degree at which maximal growth begins and persists after it provides that bound. In [12], he used this theorem as a tool to construct the Hilbert scheme for a given Hilbert function.

Theorem 2.3 (Gotzmann's Regularity Theorem). *Let X be a closed subscheme in $\mathbb{P}^N$. Then there exist positive numbers $a_s \geq a_{s-1} \geq \cdots \geq a_1 \geq 0$ such that Hilbert polynomial of X is of the form:*

(2.1)

$$P_X(n) = \binom{a_s + n}{n} + \binom{a_{s-1} + (n-1)}{n-1} + \cdots + \binom{a_1 + (n-s+1)}{n-s+1}.$$

Furthermore, I_X is s-regular.

The number of the binomial summands in (2.1) is said to be *Gotzmann number* of X. According to Gotzmann's regularity theorem, the Hilbert function of a closed subscheme X in $\mathbb{P}^N$ achieves the upper bound given by Macaulay's Theorem at large degrees. The *Gotzmann number*, denoted by $G(X)$ or $G(R/I_X)$, is defined as the smallest degree where the Hilbert function stabilizes and satisfies Macaulay's bound for all higher degrees. This is given by:

$$G(R/I_X) = \min\{d \mid H_X(k+1) = H_X(k)^{\langle k \rangle} \text{ for all } k \geq d\}. \tag{2.2}$$

For a saturated ideal I_X, it is known that $G(R/I_X) = s$ in Theorem 2.3, and generally, for non-saturated homogeneous ideal I, $G(R/I)$ is greater than or equal to s [1, Theorem 2.14].

We introduce the definition of i-th Gotzmann coefficients, which is necessary for explaining the main result of this paper ([1]). Let X be a closed subscheme in $\mathbb{P}^N$. We define the *Gotzmann coefficients* of X, denoted $C_i(X)$, as follows: with the notations in Theorem 2.3, the number of a_k which is equal to i as

$$C_i(X) = |\{k \mid a_k = i\}| \tag{2.3}$$

for all $i \geq 0$. These coefficients are uniquely determined for X and provide a decomposition of the Gotzmann number $G(X)$ into a sum:

$$G(X) = C_r(X) + C_{r-1}(X) + \cdots + C_1(X) + C_0(X). \tag{2.4}$$

We remark that $C_i(X)$, the i-th Gotzmann coefficient, is determined by the Hilbert polynomial of X and allows us to reconstruct it. Since the degree of the Hilbert polynomial is equal to the dimension of X, we have $r = \dim(X)$, and by comparing the coefficients of the leading term, note that $C_r(X)$ is equal to $\deg(X)$.

Example 2.4. *Let X be a hypersurface in $\mathbb{P}^N$ defined by a polynomial F of degree d. From the following exact sequence*

$$0 \to R(-d) \overset{\times F}{\to} R \to R/I_X \to 0$$

the Hilbert polynomial of X is given by

$$\begin{aligned} H_X(n) &= \binom{N+n}{n} - \binom{N+n-d}{n-d} \\ &= \binom{(N-1)+n}{n} + \binom{(N-1)+n-1}{n-1} + \cdots + \binom{(N-1)+n-d+1}{n-d+1}. \end{aligned}$$

Hence, we have $(C_{N-1}(X), C_{N-2}(X), \cdots, C_0(X)) = (\deg(X), 0, \cdots, 0)$.

Example 2.5. *Let X be a smooth integral curve of degree d and genus g in $\mathbb{P}^3$. Then we know that the Hilbert polynomial of X is*

$$\begin{aligned} P_X(n) &= dn + 1 - g \\ &= \binom{n+1}{n} + \cdots + \binom{n-d+2}{n-d+1} + \binom{n-d}{n-d} + \cdots + \cdots + \binom{n-\delta}{n-\delta}, \end{aligned}$$

where $\delta = d + \binom{d-1}{2} - g - 1$. Hence we have

$$(C_1(X), C_0(X)) = \left(\deg(X), \binom{d-1}{2} - g\right)$$

Note that Gotzmann coefficients contain some geometric information of X. In particular, $C_0(X)$ is exactly the same as the number of nodal points of $\pi(X)$, where $\pi : X \to \mathbb{P}^2$ is a generic projection of X to the plane.

EXAMPLE 2.6. *Let X be a sufficiently general complete intersection of $(2,2)$-type in $\mathbb{P}^4$. Using Koszul complex, we obtain $P_X(n) = 2n^2+2n+1$, which can be written by*

$$\binom{n+2}{n} + \cdots + \binom{n-1}{n-3} + \binom{n-3}{n-4} + \binom{n-4}{n-5} + \binom{n-6}{n-6} + \cdots + \binom{n-15}{n-15}.$$

Hence we have

$$(C_2(X), C_1(X), C_0(X)) = (4, 2, 6).$$

Note that $C_2(X) = \deg(X)$. What can we say about $C_1(X)$? If we consider a generic projection π: $X \to \mathbb{P}^3$, then, calling the singular locus X_1 of $\pi(X)$, its defining ideal I_{X_1} can be computed using the subresultant of the Sylvester matrix [5, Theorem 3.6]. *Calculating this with Macaulay2* [16] *reveals that X_1 is a plane conic, hence* $\deg(X_1) = 2$, *which is exactly equal to $C_1(X)$.*

Generally, Gotzmann coefficients can become very large. The following example illustrates this for us:

EXAMPLE 2.7. *Let X be the secant variety of the rational normal curve C in $\mathbb{P}^5$. We can also consider the generic projection $\pi(X)$ of X onto a hyperplane, where we find the degree of the singular locus X_1 in $\pi(X)$ to be* 7.

Meanwhile, using the Eagon-Northcott complex, we obtain the Hilbert series of X:

$$\mathbf{H}_X(t) = \frac{1+2t+3t^2}{(1-t)^4}.$$

Therefore, the Hilbert polynomial of X is

$$P_X(n) = 6\binom{n+3}{3} - 8\binom{n+2}{2} + 3\binom{n+1}{1}.$$

Then we have

$$(C_3(X), C_2(X), C_1(X), C_0(X)) = (6, 7, 46, 1382)$$

Note that $\deg(X) = C_3(X)$ *and* $\deg(X_1) = C_2(X)$.

3. MAIN RESULTS

We may expect that the Gotzmann coefficients of $X \subset \mathbb{P}^N$ contain some information on a generic projection of X to $\mathbb{P}^{N-1}$, as seen in Examples 2.5, 2.6 and 2.7.

In this section, for a non-degenerate, smooth variety $X \subset \mathbb{P}^N$ of codimension 2, we show that such information regarding the projection from a general point in $\mathbb{P}^N$ can be obtained from the Gotzmann coefficients of X using the lexicographic generic initial ideal of I_X (Theorem 3.3).

In what follows, we assume only the degree lexicographic order for monomials in $R = k[x_0, \ldots, x_N]$. To state the main result, it is necessary to introduce the concept of the i-th partial elimination ideal of I_X. This defines the set of points within the projection image, each corresponding to a fiber whose length exceeds $i > 0$.

Definition 3.1 ([5, 6]). Let I be a homogeneous ideal in $R = k[x_0, \ldots, x_n]$. For any $f \in I$, consider f as a polynomial in the variable x_0 and write it in descending order as follows:

$$f = x_0^i \bar{f} + g,$$

where $\bar{f} \in \bar{R} = k[x_1, \ldots, x_n]$ and $\deg_{x_0}(g) < i$. We then define

$$K_i(I) = \{\bar{f} \in \bar{R} \mid f \in I, \deg_{x_0}(f) \leq i\}.$$

We shall refer to $K_i(I)$ as the i-th partial elimination ideal of I, which becomes an ideal in $\bar{R} = k[x_1, \ldots, x_n]$.

Proposition 3.2 ([5, 6]). *Let $X \subset \mathbb{P}^N$ be a reduced closed subscheme and assume that $p = [1, 0, \ldots, 0] \notin X$. Consider the projection $\pi : X \to \mathbb{P}^{N-1}$ from the point $p \in \mathbb{P}^N$ to $x_0 = 0$. Then, set theoretically, $K_i(I_X)$ is the ideal of*

$$X_i = \left\{q \in \pi(X) \mid \text{length}\left(\pi^{-1}(q)\right) > i\right\}.$$

From the definition of the partial elimination ideal, we note that $K_0(I_X)$ serves as the defining ideal of the projection image $\pi(X)$, and in many cases, $K_1(I_X)$ set-theoretically defines the singular locus X_1 of $\pi(X)$. For an equi-dimensional reduced scheme X of codimension 2, it can be shown that $K_1(I_X)$ is always a saturated ideal[**?**]. This naturally prompts us to consider the scheme structure as follows:

$$Y_i = \text{Proj}\left(\bar{R}/K_i(I_X)\right), \quad \text{for } i \geq 0$$

where $\bar{R} = k[x_1, \ldots, x_n]$ is the polynomial ring obtained by eliminating the variable x_0 from R. It is important to note that $X_i = (Y_i)_{\text{red}}$. Determining when the closed subscheme Y_i becomes a reduced scheme is an interesting problem, which seems to be deeply connected with the geometric properties of X.

We are ready to give the following result, which is the main theorem in this paper.

Theorem 3.3. *Let X be a non-degenerate smooth variety of dimension $r \geq 1$ and codimension 2 in $\mathbb{P}^N$, where $N = r + 2$. Additionally, let*

$$G(X) = C_r(X) + C_{r-1}(X) + \cdots + C_1(X) + C_0(X)$$

be the decomposition of $G(X)$ into the sum of Gotzmann coefficients $C_i(X)$. Consider a projection $\pi : X \to \mathbb{P}^{r+1}$ from a general point of $\mathbb{P}^N$. Then, we have

(a) $\deg(Y_0) = C_r(X)$
(b) $\deg(Y_1) = C_{r-1}(X)$.

Proof. By the property of generic projection, the dimension and degree of $Y_0 = \pi(X)$ remain unchanged, which immediately makes (a) apparent by comparing the coefficients of the highest terms in the Hilbert polynomial. Therefore, we will focus on proving (b).

If X is a smooth curve, then, as shown in Example 2.5, the results (a) and (b) follows. Note that the scheme Y_1 defined by $K_1(I_X)$ is indeed reduced, and $C_1(X)$ is the same as the number of nodal points on the plane curve $\pi(X)$.

Considering that $\dim(X) \geq 2$ and $d = \deg(X)$, and noting that X is a smooth variety, the General Projection Theorem by Gruson and Peskine [13, Theorem 1.1] says that

$$\dim(Y_0) = r, \quad \dim(Y_1) = r - 1, \quad \dim(Y_2) = r - 2, \cdots . \tag{3.1}$$

Each dimension matches the degree of its Hilbert polynomial. Since $K_0(I_X)$ defines $\pi(X)$, the projection image, Y_0 is a hypersurface in $\mathbb{P}^{r+1}$ with dimension r and degree d.

Regarding lexicographical order, I_X has the following decomposition:

$$\mathrm{in}(I_X) = \bigoplus_{i=0}^{\infty} x_0^i\big(\mathrm{in}(K_i(I_X))\big). \tag{3.2}$$

Furthermore, assuming a generic projection, we see that $\mathrm{in}(K_i(I_X)) = \mathrm{Gin}(K_i(I_X))$ (Proposition 3.3 in [5]). Meanwhile, the Hilbert function of I_X is the same as that of its initial ideal $\mathrm{in}(I_X)$, from which we derive the following.

$$\begin{aligned}H_X(n) =& \binom{N+n}{N} - \dim_k(\mathrm{Gin}(I_X), n)\\ =& \sum_{i\geq 0}\left[\binom{N-1+n-i}{N-1} - \dim_k(\mathrm{Gin}(K_i(I_X))), n-i)\right]\\ =& \sum_{i\geq 0} H_{R/K_i(I_X)}(n-i).\end{aligned}$$

Given that the equation holds for sufficiently large n, we can express the Hilbert polynomial as follows:

$$P_X(n) = P_{Y_0}(n) + P_{Y_1}(n-1) + P_{Y_2}(n-2) + \cdots . \tag{3.3}$$

Now, since Y_0 is a hypersurface in $\mathbb{P}^{r+1}$ with dimension r and degree $d = \deg(X)$, by applying the Hilbert polynomial of Y_0 as shown in Example 2.4, we calculate:

$$\begin{aligned}P_{Y_1}(n) =& P_X(n+1) - P_{Y_0}(n+1) - P_{Y_2}(n-1) - \cdots\\ =& \underbrace{\binom{(r-1)+n+1-d}{n+1-d} + \cdots + \binom{(r-1)+n+1-d-C_{r-1}(X)+1}{n+1-d-C_{r-1}(X)+1}}_{\text{number of binomials is } C_{r-1}(X)}\\ &+ \left(\text{the sum of binomials } \binom{a}{b} \text{ with } a-b<r-1\right).\end{aligned}$$

Thus, we can write

$$P_{Y_1}(n) = \frac{C_{r-1}(X)}{(r-1)!}n^{r-1} + \ \text{(a polynomial in } n \text{ of degree} < r-1)$$

and consequently conclude that $\deg(Y_1) = C_{r-1}(X)$. $\square$

Remark 3.4. There's a subtle point in Theorem 3.3 that we need to consider. Set-theoretically, the singular locus of $\pi(X)$ is X_1, which means that generally, the Gotzmann coefficient $C_{r-1}(X)$ is not the degree of X_1, but rather gives its upper bound. It's important to note that the equality holds in the following equivalent cases:

(a) $K_1(I_X) = \sqrt{K_1(I_X)}$, that is, $K_1(I_X)$ is a reduced ideal.
(b) $C_{r-1}(X) = \deg(X_1)$.

Empirically, for varieties familiar to us, it can be calculated that $K_1(I_X)$ often becomes a reduced ideal, and thus, in many cases, $C_{r-1}(X)$ is equal to $\deg(X_1)$. It is intriguing to know when $K_1(I_X)$ becomes identical to its radical ideal. For example, whether $K_1(I_X)$ always becomes a reduced ideal when X is smooth and undergoes a generic projection is one of the open problems.

Example 3.5. *Let $X \subset \mathbb{P}^4$ be a generic projection of the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. Then, the Hilbert polynomial of X is given by*

$$P_X(n) = 2n^2 + 3n + 1.$$

We find the Gotzmann coefficients to be

$$(C_2(X), C_1(X), C_0(X)) = (4, 3, 11).$$

Taking a generic projection of X onto $\mathbb{P}^3$, the image becomes a surface $Y_0 = X_0 \subset \mathbb{P}^3$ of degree 4 (known as a Steiner surface). Computing Y_1 with Macaulay2 reveals that it is reduced and consists of three non-degenerate lines intersecting at a point. Therefore, Y_1 defines the singular locus X_1 of X_0, and its degree is 3, exactly matching $C_1(X)$. Since $P_{X_1}(n) = 3n + 1$, we have

$$P_{Y_2}(n) = P_X(n+2) - P_{X_0}(n+2) - P_{X_1}(n+1) = 1,$$

which means Y_2 is a finite scheme of degree 1. Hence, $K_2(I_X)$ is also a reduced ideal defining the singular locus of X_1.

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